CHAPTER 5

1.1e INDICES

DEFINITIONS

For $n \in \mathbb{Z}$, the number a^n means a multiplied by itself n times. a is called the base number and n the **index** (plural: indices), **power** or **exponent**. a^n can be read as a to the power n or a to the nth.

Rules

There are 3 main rules of indices, which are derived by common sense from integral indices:

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$\left(a^m\right)^n = a^{mn}$$

The application of these rules leads to the following definitions which enable the rules to be applied to any real number, not just positive integers.

$$a^{-n} = \frac{1}{a^n}$$
$$a^0 = 1$$
$$a^{1/n} = \sqrt[n]{a}$$
$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

You should be adept at applying these rules, which are essential throughout this course, and especially in calculus (module 1.3). To make sure that you can handle indices confidently here is a revision exercise.

EXERCISE 5 A

1. Evaluate the following, first ***without using your calculator***, then do some of them with your calculator to make sure you can use the index keys on your calculator properly.

(a) 8^0	(b) 2^{-1}	(c) 3^{-2}	(d) 10^{-3}	(e) $(\frac{1}{7})^{-1}$	(f) $25^{\frac{1}{2}}$
(g) $64^{\frac{1}{3}}$	(h) $8^{-\frac{1}{3}}$	(i) $27^{\frac{2}{3}}$	(j) $4^{\frac{3}{2}}$	(k) $\sqrt[3]{8^2}$	(l) $\sqrt[4]{16^3}$
(m) $4^2 + 3^2$	(n) $(4 + 3)^2$	(o) $16^{-\frac{1}{4}}$	(p) $(\frac{3}{4})^0$	(q) $(-8)^{-\frac{2}{3}}$	(r) $(2\frac{1}{4})^{\frac{1}{2}}$
(s) $(3\frac{3}{8})^{-\frac{2}{3}}$	(t) $(\frac{2}{3})^2$	(u) $(\frac{1}{4})^2$	(v) $(6\frac{1}{4})^{\frac{3}{2}}$	(w) 64^0	(x) $(\frac{3}{4})^{-1}$

2. Simplify the following:
   (a) $(x^2)^3$ (b) $(x^3)^{-1}$ (c) $(x^{\frac{1}{2}})^4$ (d) $\sqrt[3]{x^5}$ (e) $\sqrt[3]{x^6}$ (f) $\left(\sqrt[4]{x^6}\right)$

3. Find the value of x in each of the following:
   (a) $2^x = 32$ (b) $3^x = 81$ (c) $2^x = \frac{1}{4}$ (d) $3^x = 1$
   (e) $2^x = 0.5$ (f) $3^x = \frac{1}{27}$ (g) $2^5 \times 2^4 = 2^x$ (h) $2^x \times 2^3 = 2^9$
   (i) $3^x = \dfrac{3^5 \times 3^7}{3^6}$ (j) $\dfrac{2^5 \times 2^7}{2^3 \times 2^x} = 2$

FURTHER PROBLEMS INVOLVING INDICES

Type 1 using the fact that $a^x = a^y \Rightarrow x = y$
Method: reduce both sides of the equation to powers of the same base number.
(N.B. It is important to be able to recognise binary numbers (powers of 2) and powers of 3 and 5)

Example 5.1
Solve for x $4^{2x} = 8^{x+1}$ ①

- both 4 and 8 are powers of 2, so ① becomes $(2^2)^{2x} = (2^3)^{x+1}$
- simplify: $2^{4x} = 2^{3(x+1)} \rightarrow 2^{4x} = 2^{3x+3}$
- equate indices: $4x = 3x + 3 \quad \therefore x = 3$
- (check: $4^{2x} = 4^6 = 4096, \quad 8^{x+1} = 8^4 = 4096$ ✓)

Type 2 Problems that can be reduced to quadratic equations

Example 5.2
Solve for x $2^{2x} - 3(2)^{x+1} + 8 = 0$

- make a substitution: let $2^x = y$
- rewrite the terms using the substitution: $2^{2x} = y^2$ and $2^{x+1} = 2^x \times 2 = 2y$
- rewrite the equation: $y^2 - 6y + 8 = 0$
- solve for y: $(y-2)(y-4) = 0 \rightarrow y = 2$ or $y = 4$
- reconvert to x: $2^x = 2$ or $2^x = 4$
- solve for x: $x = 1$ or $x = 2$

EXERCISE 5 B

1. Solve for x
   (a) $5^x = 125$ (b) $3^{2x} = 81$ (c) $2^x = \frac{1}{4}$
   (d) $4^x = 32$ (e) $9^{x+1} = 27^x$ (f) $4^{2x} = 8^{x+1} \times 16$
   (g) $4^x = 2^{x-1} \times 8^{x+2}$ (h) $3^{-x} \times 9^{x+2} = 3$

2. Solve for x

(a) $2^{2x} - 10(2^x) + 16 = 0$ (b) $3^{2x} - 2(3^x) - 3 = 0$ (c) $2^{2x} - 2^{x+1} + 1 = 0$

(d) $2^{2x} - 9(2^x) + 8 = 0$ (e) $9^x - 4(3^x) + 3 = 0$ (f) $2^{2x+1} - 17(2^x) + 8 = 0$

(g) $4^x - 6(2^x) + 8 = 0$ (h) $4^x - 3(2^x) + 2 = 0$ (i) $9^x - 10(3^x) + 9 = 0$

THE END